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Common Fixed Point Theorem for Asymptotically Regular Mappings in Hilbert Spaces

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ABSTRACT: In this paper we prove common and coincidence fixed point theorems for asymptotically regular mapping for various contractive conditions on a Hilbert space setting. we will also study the uniqueness of solution using available data for common fixed point problem. Our result generalize several well known results in literature.

Keywords: Asymptotically regular mappings, common and coincidences fixed points, weakly compatible mappings, Hilbert spaces**.**

I. INTRODUCTIONS AND PRELIMINARIES

Most fixed point theorem for mapping in metric space satisfying different contraction condition may be extended to abstract space like Hilbert space, Banach spaces etc. After some modifications Banach fixed point theorem and its application are commonly (well) known to us.

Many authors have also extended this theorem bringing more contractive conditions somehow favours the existence of a fixed point. Almost every condition favours the asymptotic regularity of the mapping after using some considerations. So the investigation on some regular maps has a important role in fixed point theory.

Sharma and Yuel [7] and Guay and Singh [3] were among the first who used the concept of asymptotic regularity to prove fixed point theorems for wider class of mappings than a class of mappings introduced and studied by Ćirić [2].

The purpose of this paper is to prove some common and coincidences fixed point theorems in Hilbert spaces and we study the well- posedness of their fixed point problem.

Definition 1.1. A self mapping T on a closed subset of a Hilbert space H is said to be asymptotically regular at a point x in H, if

$$\| T^n x - T^{n-1} x \| \to 0 \; as \; n \to \infty$$

where $T^n x$ denotes the nth iterate of T at x.

Definition 1.2. Let C be a closed subset of a Hilbert space H. A sequence $\{x_n\}$ in C is said to be asymptotically T - regular if $\| x_n - Tx_n \| \to 0 \; as \; n \to \infty$

Definition 1.3. A pair of mappings (f, T) on a Hilbert space His said to be weakly compatible if f and T commute at their coincidence point (i.e. f T x = T f x whenever f x = T x. A point yϵH is called point of coincidence of two self – mappings f and T on Hif there exists a point xϵH such that $y = T_x = f_x$

The following lemma was given in [5] in a metric space setting.

Main Result:

Theorem 2.1: Let C be closed subset of a Hilbert space H and S and T be a mapping on C into itself satisfying

$$\| S_x - T_y \|^2 \leq \alpha \| x - S_x \|^2 + \beta \| y - T_y \|^2 + \gamma \| x - y \|^2 + \delta \, Min\{\| x - T_y \|^2, \| y - S_x \|^2\} + \eta \frac{\| y - S_x \|^2}{1 + \| x - S_x \| \, \| x - T_y \|} + \xi \frac{\| x - S_x \|^2}{1 + \| x - S_x \| \, \| x - T_y \|} \quad (1.1)$$

For all $x, y \epsilon C$, Where α, β, γ *and* δ, ξ, η are non-negative real with $\alpha + \beta + \gamma + \xi + 4\eta < 1$

Thus S and T have a unique common fixed point in C.

Proof: Let $x_0 \epsilon\, c$, we define a sequence $\{x_n\}$ as follows

$x_{2n+1} = Sx_{2n}$, $x_{2n+2} = T\, x_{2n+1},\ n = 0,1,2,3 \ldots$ from (1.2)

We have

$\| x_{2n+1} - x_{2n} \|^2 = \| s_{x_{2n}} - T_{x_{2n-1}} \|^2 \leq \alpha \| x_{2n} - s_{x_{2n}} \|^2 + \beta \| x_{2n-1} - T_{x_{2n-1}} \|^2 + \gamma \| x_{2n} - x_{2n-1} \|^2 + \delta\, Min\{\| x_{2n} - T\, x_{2n-1} \|^2 \| x_{2n-1} - s_{x_{2n}} \|^2\} + \eta \frac{\|x_{2n-1} - s_{x_{2n}}\|^2}{1+\|x_{2n} - s_{x_{2n}}\|\|x_{2n} - T_{x_{2n-1}}\|} + \xi \frac{\|x_{2n} - s_{x_{2n+1}}\|^2}{1+\|x_{2n-1} - s_{x_{2n}}\|\|x_{2n} - T_{x_{2n-1}}\|}$

$\leq \alpha \| x_{2n} - x_{2n+1} \|^2 + \beta \| x_{2n-1} - x_{2n} \|^2 + \gamma \| x_{2n} - x_{2n-1} \|^2 + \delta\, Min\{\| x_{2n} - x_{2n} \|^2, \| x_{2n-1} - x_{2n+1} \|^2\}$
$+ \eta \frac{\|x_{2n-1} - x_{2n+1}\|^2}{1+\|x_{2n} - x_{2n+1}\|\|x_{2n} - x_{2n}\|} + \xi \frac{\|x_{2n} - s_{x_{2n+1}}\|^2}{1+\|x_{2n-1} - x_{2n+1}\|^2\|x_{2n} - x_{2n}\|}$

Now

$(1-\alpha - \xi) \| x_{2n} - x_{2n+1} \|^2$

$(1-\alpha - \xi - 2\eta) \leq \beta + \gamma + 2\eta \| x_{2n} - x_{2n+1} \|^2$

$$\| x_{2n} - x_{2n+1} \|^2 \leq \frac{\beta + \gamma + 2\eta}{(1-\alpha - \xi - 2\eta)}$$

Putting $q = \frac{\beta+\gamma+2\eta}{(1-\alpha-\xi-2\eta)} < 1$

Then we have:

$$\| x_{2n} - x_{2n+1} \|^2 \leq q \| x_{2n} - x_{2n-1} \|^2$$

Processing in this way :

$$\| x_{2n} - x_{2n+1} \|^2 \leq q^n \| x_{2n} - x_{2n-1} \|^2$$

For any positive integer p, one gets:

$$\| x_n - x_{n+p} \|^2 \leq \| q x_n - x_{n+1} \| + \| x_{n+1} - x_{n+2} \| + \cdots + \leq (q^n + q^{n+1} + q^{n+2} + \cdots + q^{n+p-1}) \| x_0 - x_1 \|$$

$$\| x_n - x_{n+p} \| \leq \frac{k^n}{1-k} \| x_0 - x_1 \|$$

Thus $\| x_n - x_{n+p} \| \to 0$ as $n \to \infty$

Hence $|x_n|$ is a Cauchy sequence in C. Since C is closed subset Of H, then There exists an element $v \in C$ such that

$$\lim_{n\to\infty} x_n = x$$

Now further, we have

$\| v - T_v \|^2 = (\| v - x_{2n+1} \| + \| x_{2n+1} - T_v \|)^2$

$\leq \| v - x_{2n+1} \|^2 + \| x_{2n+1} - T_v \|^2 + 2\mathrm{Re}\| v - x_{2n+1} \| + \| x_{2n+1} - T_v \|$

$\leq \| v - x_{2n+1} \|^2 + \alpha \| x_{2n} - x_{2n+1} \|^2 + \beta \| v - T_v \|^2 + \gamma \| x_{2n} - v \|^2 + \delta\{\| x_{2n} - T_v \|^2, \| v - x_{2n+1} \|^2\} +$
$\eta \frac{\|v - x_{2n+1}\|^2}{1+\|x_{2n} - x_{2n+1}\| + \|x_{2n} - T_v\|} + \xi \frac{\|x_{2n} - x_{2n+1}\|^2}{\|v - x_{2n+1}\| + \|x_{2n} - T_v\|} + 2\,\mathrm{Re} < v - x_{2n+1}, x_{2n+1} - T_v >$

As $n \to \infty, x_{2n} \to v, x_{2n+1} \to v$

We have

$2\,\mathrm{Re} < v - x_{2n+1}, x_{2n+1} - T_v > \to 0$

Then $\| v - T_v \|^2 \leq \beta \| v - T_v \|^2$

Then implies that v, T_v, Since $\beta < 1$ Similarly we get $v = s_v$, then v is a common fixed point of S and T.

For the uniqueness Let $u \in c$ be another fixed point S and T. When $u \neq v$

$$\| u - v \|^2 = \| s_v - T_u \|^2$$

$\alpha \| v - s_v \|^2 + \beta \| u - T_u \|^2 + \gamma \| u - v \|^2 + \delta Min\{\| v - T_u \|^2, \| u - s_v \|^2\} +$
$\eta \frac{\|u - s_v\|^2}{1+\|v - s_v\|\|v - T_u\|} + \xi \frac{\|v - s_v\|}{1+\|u - s_v\|\|v - T_u\|} \leq (v + \delta + \eta) \| u - v \|^2$

Since $\gamma + \delta + \eta < 1$

$so\ u = v$ that is the common fixed point is unique.

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